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Correlation functions of higher-dimensional automatic sequences

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Abstract

A procedure for calculating the (auto)correlation function $\gamma_f(k)$, $k \in \mathbb{Z}^m$, of an m-dimensional complex-valued automatic sequence $f : \mathbb{Z}^m \to \mathbb{C}$, is presented. This is done by deriving a recursion for the vector correlation function $\Gamma_{\ker(f)}(k)$ whose components are the (cross)correlation functions between all sequences in the finite set $\ker(f)$, the so-called kernel of f which contains all properly defined decimations of f. The existence of $\Gamma_{\ker(f)}(k)$, which is defined as a limit, for all $k \in \mathbb{Z}^m$, is shown to depend only on the existence of $\Gamma_{\ker(f)}(0)$. This is illustrated for the higher-dimensional Thue–Morse, paper folding and Rudin–Shapiro sequences.

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1. Introduction and preliminaries

Automatic sequences $f : \mathbb{Z}^m \to \mathbb{C}$, where $\mathbb{C}$ denotes the complex numbers, are characterized by the fact that they have a finite number of properly defined decimations. Equivalently, they are generated by a fixed point of a certain substitution system, or by a particular finite automaton (see [1], [2], and the extended list of references therein). In [3], we discussed the conditions under which automatic sets (a particular form of automatic sequences) are Delone sets. The diffraction spectrum of Delone sets is typically a mixture of a pure point spectrum (typical for quasi-periodic sets) and a continuous spectrum (either absolute or singular continuous), and is given by the Fourier transform of the (auto)correlation function of the underlying set. The spectral properties thus depend on this correlation function, and hence knowledge of the correlation function is a first step towards characterizing the spectrum, see e.g., [4] and [5] for further details.

For a complex-valued sequence $f : \mathbb{Z}^m \to \mathbb{C}$, the (auto)correlation function is usually defined, provided the limit exists, as

$$\gamma_{ff}(k) = \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R(0))} \sum_{x\in B_R(0)\cap\mathbb{Z}^m} f(x)\overline{f}(x+k),$$

for all $k \in \mathbb{Z}^m$, where $B_R(0)$ is a ball of radius R in $\mathbb{R}^m$ (in a proper norm) and $\overline{f}$ denotes the complex conjugate of f. Correlation and spectral properties of special one-dimensional automatic sequences (over $\mathbb{N}$) such as the Thue–Morse, paper folding and Rudin–Shapiro sequences have been studied in part IV of [6], where also further references can be found. Since one-dimensional automatic sequences can also be considered as generated by a substitution of constant length, the book [7] provides a further source of information on spectral properties of sequences. Correlation and spectral properties of a certain class of higher dimensional substitution sequences are studied in [8, 9].

In this paper, we present a procedure for calculating the correlation function $\gamma_{ff}(k)$, $k \in \mathbb{Z}^m$ of an m-dimensional complex-valued H-automatic sequence $f : \mathbb{Z}^m \to \mathbb{C}$, where H is a proper expanding integer matrix for which a digit set W exists such that (H, W) defines a numbering system for $\mathbb{Z}^m$ [2, 3]. This is done by deriving a recursion for the whole set of correlation functions

$$\gamma_{gh}(k) = \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R(0))} \sum_{x\in B_R(0)\cap\mathbb{Z}^m} g(x)\overline{h}(x+k),$$

where $g, h \in \ker(f)$, and $\ker(f)$ is the finite set of all properly defined decimations of f. Roughly speaking (and omitting some technical details at this stage) the main result, theorem 3.4, states the following: if $\gamma_{gh}(0)$ exists for all $g, h \in \ker(f)$, then f has a unique correlation function. Actually, we shall establish a more general result by showing that the correlations exist and are unchanged if $B_R(0)$ is replaced by a cylinder all axes of which go simultaneously and independently to infinity.

In the rest of this section, we recall the essentials about H-automatic sequences that are necessary for our purposes. For a more extended and more general treatment, see [2]. Section 2 deals with properties of shifts and products of automatic sequences. In section 3, the main result concerning the existence and calculation of correlation functions is derived. This result is given in terms of the characteristics of the graph describing the underlying automaton that generates the sequence, and complements conditions for the existence of correlation functions in terms of substitutions, cf [8, 9]. The result is illustrated for a whole class of higher-dimensional Thue–Morse, paper folding and Rudin–Shapiro sequences. A follow-up paper will deal with the correlation and spectral properties for these particular sequences. These sequences are straightforward generalizations, as described in [3], of the corresponding original one-dimensional 2-automatic sequences on $\mathbb{N}$ (see a.o. [6]) by considering the same automaton but using binary numbering systems in $\mathbb{Z}^d$ instead of in $\mathbb{N}$. They differ from other kinds of generalizations in higher dimensions which are obtained from substitutions in $\mathbb{N}^d$ ([8, 9]) or from products of one-dimensional sequences (see e.g. [10]). In the appendices, we collect some technical results necessary for the proof of the main theorem.

The definition of H-automatic sequences defined on $\mathbb{Z}^m$ requires an *expanding integer matrix* $H \in \mathbb{Z}^{m\times m}$ (expanding means that all eigenvalues have absolute value greater than 1). According to [11], then there exists a norm $\|\,\|$ and a $c > 0$ such that

$$\|Hx\| \geqslant c\|x\| \qquad \text{for all} \quad x \in \mathbb{Z}^m. \tag{1}$$

Moreover, a *complete residue set* W for H is needed: this is a set $W = \{w_0 = 0, w_1, \ldots, w_{|\det(H)|-1}\} \in \mathbb{Z}^m$ such that for every $x \in \mathbb{Z}^m$ there exist unique $\zeta(x) \in W$ and $\kappa(x) \in \mathbb{Z}^m$ such that

$$x = H\kappa(x) + \zeta(x). \tag{2}$$

A complete residue set is called *complete digit set* if for every $x \in \mathbb{Z}^m$ there exists $n = n(x) \in \mathbb{N}$ such that $\kappa^n(x) = 0$.

This is equivalent to: every $x \in \mathbb{Z}^m \backslash \{0\}$ has a finite (H, W)-*representation*, i.e., for every $x \in \mathbb{Z}^m \backslash \{0\}$ there exist unique $\omega_i \in W, i = 1, \ldots, n$ such that

$$x = H^{n-1}\omega_n + H^{n-2}\omega_{n-1} + \cdots + H\omega_2 + \omega_1,$$

and $\omega_n \neq 0$. From now on, we always assume that W is a complete digit set of H. The existence of complete digit sets for a given H is a difficult question; partial answers and related literature can be found in [12].

An *m-dimensional sequence* f is defined as a map $f : \mathbb{Z}^m \to \mathbb{C}$. The (H, w)-*decimation* of f, with $w \in W$, is defined as the sequence $\partial_w(f)$ satisfying

$$\partial_w(f)(x) = f(Hx + w). \tag{3}$$

The maps $\partial_w : \mathbb{C}^{\mathbb{Z}^m} \to \mathbb{C}^{\mathbb{Z}^m}, w \in W$, are called decimations. Repeated application of decimations to a sequence f is written as

$$\partial_{\omega_n} \circ \partial_{\omega_{n-1}} \cdots \circ \partial_{\omega_1}(f)(x) = f(H^n x + H^{n-1}\omega_n + H^{n-2}\omega_{n-1} + \cdots + \omega_1).$$

The set of all decimations of f together with f forms the (H, W)-*kernel* of f:

$$\ker_{(H,W)}(f) = \{f\} \cup \left\{\partial_{\omega_n} \circ \partial_{\omega_{n-1}} \cdots \circ \partial_{\omega_0}(f) \;\middle|\; n \in \mathbb{N}, \omega_i \in W, i = 0, \ldots, n\right\}.$$

We simply write $\ker(f)$ if H and W are clear from the context.

Definition 1.1. *The sequence* $f \in \mathbb{C}^{\mathbb{Z}^m}$ *is called* (H, W)-*automatic if* $\ker(f)$ *is finite.*

According to theorem 3.2.2 in [2], the automaticity does not depend on the choice of the residue set W. It is therefore justified to speak of an H-automatic sequence.

We now recall the fact that an automatic sequence f is also related to a fixed point of a substitution map Σ_f defined on the vector sequence $\mathbf{F} : \mathbb{Z}^m \to \mathbb{C}^{\ker(f)}$ (components labelled by the elements of $\ker(f)$). To this end, we have to define decimation matrices for a finite *decimation invariant set* $K \subset \mathbb{C}^{\mathbb{Z}^m}$. This is a set K satisfying

$$\partial_w(K) = \{\partial_w(g) \mid g \in K\} \subseteq K$$

for all $w \in W$. The *decimation matrices* for a decimation invariant set K are matrices $A_w^{[K]} = \left(a_{g,h}^w\right) \in \{0, 1\}^{K \times K}, w \in W$, defined by

$$a_{g,h}^w = \begin{cases} 1 & \text{if } \partial_w(g) = h \\ 0 & \text{otherwise.} \end{cases}$$

Note that decimation matrices have precisely a single 1 in each row. As a consequence, products and sums of decimation matrices are nonnegative matrices such that every row contains at least one positive entry. Clearly, if matrices of this type are multiplied by positive constants, this property remains. This simple observation will be of crucial importance later on.

The decimation matrices of $\ker(f)$, which is by definition decimation invariant, will be denoted by A_w or A_w^f instead of $A_w^{[\ker(f)]}$ if the context is clear.

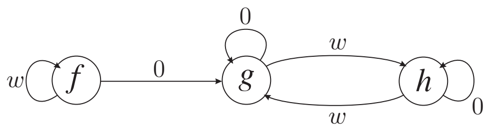

Figure 1. A kernel graph.

The substitution map Σ_f transforms the sequence $\mathbf{F} : \mathbb{Z}^m \to \mathbb{C}^{\ker(f)}$ into the sequence $\Sigma_f(\mathbf{F}) : \mathbb{Z}^m \to \mathbb{C}^{\ker(f)}$ by putting

$$\Sigma_f(\mathbf{F})(Hx + w) = A_w \mathbf{F}(x) \tag{4}$$

for all $w \in W$ and $x \in \mathbb{Z}^m$. Then the sequence $\mathcal{F} : \mathbb{Z}^m \to \mathbb{C}^{\ker(f)}$ defined as

$$\mathcal{F}(x) = (g(x))_{g \in \ker(f)}$$

for $x \in \mathbb{Z}^m$ is a fixed point of the substitution Σ_f, i.e., $\Sigma_f(\mathcal{F}) = \mathcal{F}$, see e.g., [2].

As a consequence of (4), if $x = \sum_{j=1}^{n} H^{j-1}\omega_j$ is the unique (H, W)-representation of $x \in \mathbb{Z}^m \backslash \{0\}$, then

$$\mathcal{F}(x) = A_{\omega_1} A_{\omega_2} \ldots A_{\omega_n} \mathcal{F}(0). \tag{5}$$

This implies that $\mathcal{F}(0) = A_0 \mathcal{F}(0)$. The decimation matrices A_w, $w \in W$ also define a directed graph, the *kernel graph*, where the vertices correspond to the elements of $\ker(f)$, and where a vertex $g \in \ker(f)$ is connected to a vertex $h \in \ker(f)$ by a directed edge with label w, if and only if $h = \partial_w(g)$, i.e., if $a^w_{g,h} = 1$.

For example, assume that f is an automatic $(H, \{0, w\})$-sequence, with $\ker(f) = \{f, g, h\}$ and decimation matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where the rows and columns of the matrices correspond to the elements f, g, h, in that order. The kernel graph associated with A_0, A_w is given in figure 1.

A kernel graph can be interpreted as a finite automaton that generates the sequence f, see e.g., [2]. The idea of generating a sequence is as follows: if $x \in \mathbb{Z}^m$, $x \neq 0$, has the (H, W)-representation

$$x = \sum_{j=1}^{n} H^{j-1}\omega_j,$$

then x defines a path in the directed graph. The path begins in f, follows the arrows labelled $\omega_1, \omega_2, \ldots, \omega_n$ and terminates in an element $g \in \ker(f)$. Then the value of f at x is equal to the value of g at 0, i.e., $f(x) = g(0)$.

Due to this interpretation, one immediately sees that an H-automatic sequence has only finitely many values, i.e., the set $\{f(k) \mid k \in \mathbb{Z}^m\}$ is finite.

2. Shifts and products of complex-valued automatic sequences

We will ultimately deal with convolutions of automatic sequences $f \in \mathbb{C}^{\mathbb{Z}^m}$, which involve shifts, products and sums of sequences.

First we begin with the shift of a sequence f. For $k \in \mathbb{Z}^m$, the shift map $\sigma_k : \mathbb{C}^{\mathbb{Z}^m} \to \mathbb{C}^{\mathbb{Z}^m}$ is defined as

$$\sigma_k(f)(x) = f(x + k).$$

By theorem 3.2.5 in [2], H-automaticity of f implies H-automaticity of $\sigma_k(f)$. We state the following relationship for decimations of a shift of a sequence:

Lemma 2.1.

$$\partial_w \circ \sigma_k = \sigma_{\kappa(w+k)} \circ \partial_{\zeta(w+k)}. \tag{6}$$

Proof. Let $g(x) = \sigma_k(f)(x) = f(x+k)$. Then, by invoking (3), we get $(\partial_w \circ \sigma_k)(f)(y) = \partial_w(g)(y) = g(Hy+w) = f(Hy+w+k) = f(H(y+\kappa(w+k)) + \zeta(w+k)) = \partial_{\zeta(w+k)}(f)(y+\kappa(w+k)) = \sigma_{\kappa(w+k)}(\partial_{\zeta(w+k)}(f))(y)$. □

As a next step, we recall the notion of Kronecker (tensor, direct) product of matrices and vectors, but slightly adapted to our purposes.

Let f and g be automatic sequences with respective kernels $\ker(f)$ and $\ker(g)$. If $A \in \mathbb{C}^{\ker(f)\times\ker(f)}$ and $B \in \mathbb{C}^{\ker(g)\times\ker(g)}$ are matrices, then the matrix $A \otimes B$ is an element of $\mathbb{C}^{(\ker(f)\times\ker(g))\times(\ker(f)\times\ker(g))}$ defined as

$$(A \otimes B)((f_1, g_1), (f_2, g_2)) = A(f_1, f_2)B(g_1, g_2).$$

Note further that, if $F \in \mathbb{C}^{\ker(f)}$ and $G \in \mathbb{C}^{\ker(g)}$ are vectors, then the vector $F \otimes G \in \mathbb{C}^{(\ker(f)\times\ker(g))}$, i.e., a vector whose components are labelled by the elements of $\ker(f) \times \ker(g)$, is defined as

$$(F \otimes G)(f_1, g_1) = F(f_1)G(g_1).$$

And with this, one has that the product of the matrix $A \otimes B$ with the vector $F \otimes G$ is the vector

$$(A \otimes B)(F \otimes G) = (AF) \otimes (BG),$$

where AF is the usual product of a $\ker(f) \times \ker(f)$-matrix with a $\ker(f)$-vector (similar for BG).

For the automatic sequences $f, g \in \mathbb{C}^{\mathbb{Z}^m}$, we define the vector sequence $\mathcal{F} \otimes \mathcal{G} : \mathbb{Z}^m \to \mathbb{C}^{(\ker(f)\times\ker(g))}$ by setting

$$(\mathcal{F} \otimes \mathcal{G})(x)(f_1, g_1) = f_1(x)g_1(x), \tag{7}$$

where $f_1 \in \ker(f)$, $g_1 \in \ker(g)$ and $x \in \mathbb{Z}^m$. Moreover, for matrices $A \in \mathbb{C}^{\ker(f)\times\ker(f)}$, $B \in \mathbb{C}^{\ker(g)\times\ker(g)}$ we define the vector sequence $(A \otimes B)(\mathcal{F} \otimes \mathcal{G}) : \mathbb{Z}^m \to \mathbb{C}^{(\ker(f)\times\ker(g))}$ by

$$(A \otimes B)(\mathcal{F} \otimes \mathcal{G})(x) = (A\mathcal{F}(x)) \otimes (B\mathcal{G}(x)), \tag{8}$$

where $A\mathcal{F}(x)$ and $B\mathcal{G}(x)$ are the usual matrix-vector products.

Let f and $g \in \mathbb{C}^{\mathbb{Z}^m}$ be H-automatic sequences, then the set $\ker(f) \times \ker(g)$ is decimation invariant, where the decimations of $(\phi, \psi) \in \ker(f) \times \ker(g)$ are defined componentwise, i.e., $\partial_w(\phi, \psi) = (\partial_w(\phi), \partial_w(\psi))$. It is therefore meaningful to consider the decimation matrices of the decimation invariant set $\ker(f) \times \ker(g)$.

Lemma 2.2. *If f and g are H-automatic sequences, then*

$$A_w^{[\ker(f)\times\ker(g)]} = A_w^{[\ker(f)]} \otimes A_w^{[\ker(g)]}$$

for all $w \in W$.

Proof. By the definition of the tensor product

$$\left(A_w^f \otimes A_w^g\right)((f_1, g_1), (f_2, g_2)) = A_w^f(f_1, f_2)A_w^g(g_1, g_2)$$

is equal to 1 if and only if $A_w^f(f_1, f_2) = 1$ and $A_w^g(g_1, g_2) = 1$. On the other hand, by the definition of the decimation matrix

$$A_w^{[\ker(f)\times\ker(g)]}((f_1, g_1), (f_2, g_2)) = 1$$

if and only if $\partial_w(f_1) = f_2$ and $\partial_w(g_1) = g_2$, which is equivalent to $A_w^f(f_1, f_2) = 1$ and $A_w^g(g_1, g_2) = 1$. This completes the proof. □

If $\rho : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is any map and if f and g are H-automatic sequences, then $h : \mathbb{Z}^m \to \mathbb{C}$ defined as $h(x) = \rho(f(x), g(x))$ is H-automatic. Indeed, $\partial_w(h)(x) = h(Hx + w) = \rho(f(Hx + w), g(Hx + w)) = \rho(\partial_w(f)(x), \partial_w(g)(x))$. Then the decimation matrices A_w^h are submatrices of the matrices $A_w^{[\ker(f)\times\ker(g)]}$.

Example

(i) Consider $h = \rho(f, f)$, then the decimation matrices A_w^h are the submatrices of $A_w^{[\ker(f)\times\ker(f)]}$ formed by the entries $((f_1, f_1), (f_2, f_2))$. Except for the labelling of the entries, this matrix is identical to $A_w^{[\ker(f)]}$ itself.

(ii) Consider the so-called two-dimensional Thue–Morse sequence t and the paper folding sequence p as introduced in [3]. These sequences are H-automatic sequences w.r.t. some proper H. For a complete digit set $W = \{0, w\}$, the respective decimation matrices are given as

$$A_0^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_w^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$A_0^p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_w^p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $h(x) = t(x) + p(x)$ is H-automatic, and one computes that indeed

$$A_v^h = A_v^t \otimes A_v^p \qquad \text{for} \quad v = 0, w_1.$$

Another consequence of lemma 2.2 in combination with equations (7) and (8) is the following relation:

$$(\mathcal{F} \otimes \mathcal{G})(Hx + w) = \left(A_w^f \otimes A_w^g\right)((\mathcal{F} \otimes \mathcal{G})(x)). \tag{9}$$

3. Convolutions and correlation functions

From now on, we suppose that the expanding matrix H satisfies an additional condition: there exists a matrix $P \in \mathbb{R}^{m\times m}$ such that $P^{-1}HP$ is an $m \times m$ block-diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s, \Lambda_1, \Lambda_2, \ldots, \Lambda_t), \tag{10}$$

where the λ_j, $|\lambda_j| > 1$, correspond to the real eigenvalues of H, and the Λ_j are 2×2-matrices of the form

$$\Lambda_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \tag{11}$$

where $a_j, b_j \in \mathbb{R}$ and $|\det(\Lambda_j)| = a_j^2 + b_j^2 > 1$. Λ_j corresponds to a pair of complex eigenvalues $(a_j \pm b_j i)$ of H.

If the elements of $\mathbb{R}^m$ are denoted as $\mathbf{x} = (z_1, \ldots, z_s, x_1, y_1, x_2, y_2, \ldots, x_t, y_t)$, then it is easy to see that

$$\|\mathbf{x}\|_\infty = \max\left\{\max\{|z_i| \mid i = 1, \ldots, s\}, \max\left\{\sqrt{x_i^2 + y_i^2} \,\middle|\, i = 1, \ldots, t\right\}\right\}$$

defines a norm on $\mathbb{R}^m$. Moreover, the map $\Lambda : \mathbb{R}^m \to \mathbb{R}^m$ is expanding w.r.t. this norm. Then $H : \mathbb{R}^m \to \mathbb{R}^m$ is expanding w.r.t. the norm

$$\|\mathbf{x}\|_H = \|P^{-1}\mathbf{x}\|_\infty. \tag{12}$$

The balls $B_R(0)$ appearing in the definition of the correlation function in the introduction are considered with respect to this norm. They actually correspond to the cylinders $P\mathcal{C}(R, R, \ldots, R)$ defined in appendix B, equation (B.1).

However, for computing the correlations, we will even consider cylinders $\mathcal{C}(\underline{R}) = \mathcal{C}(R_1, R_2, \ldots, R_{s+t})$ with possibly different sizes R_i in each 'direction'. The reason for this is that it allows us to formulate the next theorem, which is crucial for the further development. If f and g are H-automatic, then the convolution of f and g (of size $\underline{R}$) is the sequence $c_{\underline{R}} : \mathbb{Z}^m \to \mathbb{C}$ defined as

$$c_{\underline{R}}(f, g)(k) = \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} f(x)\overline{g}(x + k) = \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} (f\sigma_k(\overline{g}))(x),$$

where $\overline{g}$ denotes the complex conjugate. We consider the set of all convolutions between the kernel elements of f and g, i.e., all sequences $c_{\underline{R}}(\phi, \psi)$ with $\phi \in \ker(f)$ and $\psi \in \ker(g)$. Using the tensor product, we can write the total of these sequences as a $(\ker(f) \times \ker(g))$-vector

$$C_{\underline{R}}(\mathcal{F}, \mathcal{G})(k) = \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} (\mathcal{F} \otimes \sigma_k(\overline{\mathcal{G}}))(x). \tag{13}$$

The next theorem provides a kind of recursive relation for the convolutions which is a preform of the recursion involving the correlations.

Theorem 3.1. *Let $f, g \in \mathbb{C}^{\mathbb{Z}^m}$ be H-automatic and $\underline{R}/\underline{c}$ and $o_{\mathcal{C}}(\underline{R})$ be as defined in appendix B equation (B.4), then*

$$C_{\underline{R}}(\mathcal{F}, \mathcal{G})(Hk + w) = \sum_{v \in W} \left(A_v^f \otimes A_{\zeta(w+v)}^g\right) C_{\underline{R}/\underline{c}}(\mathcal{F}, \mathcal{G})(k + \kappa(w + v)) + o_{\mathcal{C}}(\underline{R}) \tag{14}$$

holds for all $w \in W$ and all $k \in \mathbb{Z}^m$.

Proof. We have by equation (13) and by lemma 3.10

$$\begin{aligned} C_{\underline{R}}(\mathcal{F}, \mathcal{G})(Hk + w) &= \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} (\mathcal{F} \otimes \sigma_{Hk+w}(\overline{\mathcal{G}}))(x) \\ &= \sum_{v \in W} \sum_{x \in P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m} (\mathcal{F} \otimes \sigma_{Hk+w}(\overline{\mathcal{G}}))(Hx + v) + o_{\mathcal{C}}(\underline{R}) \end{aligned}$$

which is, due to equations (6) and (9), the same as

$$\sum_{v \in W} \sum_{x \in P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m} \left(A_v^f \otimes A_{\zeta(w+v)}^g\right)(\mathcal{F} \otimes \sigma_{k+\kappa(w+v)}(\overline{\mathcal{G}}))(x) + o_{\mathcal{C}}(\underline{R})$$

and which, by equation (13), yields the desired expression (14).

$$\sum_{v \in W} \left(A_v^f \otimes A_{\zeta(w+v)}^g\right) C_{\underline{R}/\underline{c}}(\mathcal{F}, \mathcal{G})(k + \kappa(w + v)) + o_{\mathcal{C}}(\underline{R}). \qquad \square$$

From now on we restrict our attention to the '(auto)convolution' case $f = g$ and write $C_{\underline{R}}(k)$ instead of $C_{\underline{R}}(\mathcal{F}, \mathcal{F})(k)$. Note, for later reference, that if $k = 0$ and $w = 0$, equation (14) for $f = g$ reads

$$C_{\underline{R}}(0) = \left(\sum_{v \in W} A_v \otimes A_v\right) C_{\underline{R}/\underline{c}}(0) + o_{\mathcal{C}}(\underline{R}). \tag{15}$$

The next lemma presents some consequences of the simple observation that decimation matrices have precisely a single 1 in every row. These will be important later.

Lemma 3.2. *Let $f \in \mathbb{C}^{\mathbb{Z}^m}$ be H-automatic with decimation matrices A_v, $v \in W$.*

(i) For all $g \in \ker(f)$

$$\sum_{h\in\ker(f)} \sum_{v\in W} A_v(g,h) = |\det(H)|\,.$$

(ii) Let $\xi : W \to W$ be any map. For all $(g_1, g_2) \in \ker(f) \times \ker(f)$ and all $v \in W$

$$\sum_{h_1,h_2\in\ker(f)} \sum_{v\in W} (A_v \otimes A_{\xi(v)})((g_1, g_2), (h_1, h_2)) = |\det(H)|\,.$$

Proof.

Ad (i). Follows from the fact that every row of A_v contains a single 1 and that $|W| = |\det(H)|$.

Ad (ii). By definition

$$\sum_{h_1,h_2\in\ker(f)} \sum_{v\in W} (A_v \otimes A_{\xi(v)})((g_1, g_2), (h_1, h_2)) = \sum_{h_1,h_2\in\ker(f)} \sum_{v\in W} A_v(g_1, h_1)A_{\xi(v)}(g_2, h_2),$$

which is the same as

$$\sum_{h_1\in\ker(f)} \sum_{v\in W} A_v(g_1, h_1) \left(\sum_{h_2\in\ker(f)} A_{\xi(v)}(g_2, h_2) \right).$$

Since $A_{\xi(v)}$ is a decimation matrix, the sum over h_2 is equal to 1, and this reduces the sum to

$$\sum_{h_1\in\ker(f)} \sum_{v\in W} A_v(g_1, h_1),$$

which is equal to $|\det(H)|$, by (i). □

Instead of the convolution sequence $C_{\underline{R}}(k)$, we now consider the scaled sequence

$$\Gamma_{\underline{R}}(k) = \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} C_{\underline{R}}(k),$$

the limit of which, for $\underline{R} \Rightarrow \infty$ (meaning that all components R_i of $\underline{R}$ go simultaneously to ∞), if it exists, will give the correlation $\Gamma(k)$, a $(\ker(f) \times \ker(f))$-vector with components $\gamma_{gh}(k)$, where $g, h \in \ker(f)$. Note that, due to equation (B.2), we have

$$\mathrm{vol}(\mathcal{C}(\underline{R})) = |\det(H)|\, \mathrm{vol}(\mathcal{C}(\underline{R}/\underline{c})) \tag{16}$$

and therefore the rescaled version of theorem 3.1 reads

$$\Gamma_{\underline{R}}(Hk + w) = \frac{1}{|\det(H)|} \sum_{v\in W} (A_v \otimes A_{\zeta(w+v)})\Gamma_{\underline{R}/\underline{c}}(k + \kappa(w + v)) + \frac{o_{\mathcal{C}}(\underline{R})}{\mathrm{vol}(\mathcal{C}(\underline{R}))}. \tag{17}$$

From now on, the rightmost term in the above expression will be written as $\epsilon(\underline{R})$. By definition (see appendix B, equation (B.4)), $\lim_{\underline{R}\Rightarrow\infty} \epsilon(\underline{R}) = 0$. Under the assumption that the limits $\Gamma(k)$, $k \in \mathbb{Z}^m$, exist, equation (17) reads

$$\Gamma(Hk + w) = \frac{1}{|\det(H)|} \sum_{v\in W} (A_v \otimes A_{\zeta(w+v)})\Gamma(k + \kappa(w + v)). \tag{18}$$

Particular case. If $|\det(H)| = 2$ and if $W = \{0, w\}$ denotes a complete digit set of H, then equation (18) reads

$$\Gamma(Hk) = \tfrac{1}{2}(A_0 \otimes A_0 + A_w \otimes A_w)\Gamma(k) \tag{19}$$

$$\Gamma(Hk + w) = \tfrac{1}{2}(A_0 \otimes A_w)\Gamma(k) + \tfrac{1}{2}(A_w \otimes A_0)\Gamma(k + \hat{w}), \tag{20}$$

where $\hat{w} = H^{-1}(2w)$.

If, in equation (18), $Hk + w$ is replaced by k, this formula transforms into

$$\Gamma(k) = \frac{1}{|\det(H)|} \sum_{v \in W} (A_v \otimes A_{\zeta(\zeta(k)+v)})\Gamma(\kappa(k) + \kappa(\zeta(k) + v)). \tag{21}$$

Let us assume for the moment that the limits $\Gamma(k)$ exist for all $k \in B_r(0) \cap \mathbb{Z}^m$ and for r sufficiently large, where $B_r(0)$ is defined for a norm for which H is expanding and not necessarily the norm defined in (12). The latter norm is only necessary in the determination of the correlations as limits. Then equation (18) provides a recursive method to compute $\Gamma(k)$ for $k \in \mathbb{Z}^m$ outside $B_r(0)$. The following lemma provides a lower bound for the radius r, in the sense that a knowledge of $\Gamma(k)$, $\|k\| \leqslant r^*$, already determines the remaining values of Γ.

Lemma 3.3. *Let H be an expanding matrix with expansion constant c (see (1)), W be a related complete digit set with corresponding κ and ζ-maps (see equation (2)). Let $\alpha = \max\{\|\kappa(v_1 + v_2)\| \mid v_1, v_2 \in W\}$, $\beta = \max\{\|v\| \mid v \in W\}$ and*

$$r^* = \frac{\beta + c\alpha}{c - 1}. \tag{22}$$

(i) If $k \in \mathbb{Z}^m$ is such that $\|k\| > r^*$, then

$$\|\kappa(k) + \kappa(\zeta(k) + v)\| < \|k\|$$

for all $v \in V$.

(ii) If k is such that $\|k\| \leqslant r^*$, then

$$\|\kappa(k) + \kappa(\zeta(k) + v)\| \leqslant r^*$$

for all $v \in V$.

Proof. Set $f_v(k) = \kappa(k) + \kappa(\zeta(k) + v)$ for $v \in W$ and $k \in \mathbb{Z}^m$, then

$$\|f_v(k)\| \leqslant \|\kappa(k)\| + \alpha,$$

where $\alpha = \max\{\|\kappa(v_1 + v_2)\| \mid v_1, v_2 \in W\}$. Now note that

$$\|k - \zeta(k)\| = \|H\kappa(k)\| \geqslant c\|\kappa(k)\|.$$

This gives

$$\|\kappa(k)\| \leqslant \frac{\|k\| + \beta}{c}$$

for $\beta = \max\{\|v\| \mid v \in W\}$ and therefore

$$\|f_v(k)\| \leqslant \frac{\|k\| + \beta + c\alpha}{c}.$$

Using (22), one obtains that $\|f_v(k)\| < \|k\|$ if $\|k\| > r^*$ and $\|f_v(k)\| \leqslant r^*$ if $k \leqslant r^*$. $\square$

In other words, referring to the recursion (21), knowledge of $\Gamma(\kappa(k) + \kappa(\zeta(k) + v))$, $v \in W$, allows a computation of $\Gamma(k)$. Indeed, if $\|k\| > r^*$, then $\|\kappa(k) + \kappa(\zeta(k) + v)\| < \|k\|$ for all $v \in W$. Therefore, a knowledge of $\Gamma(k)$ for $k \in B_{r^*}(0)$ is sufficient to compute $\Gamma(k)$ for the remaining values of k outside $B_{r^*}(0)$. It follows of course that the mere existence of $\Gamma(k)$ in $B_{r^*}(0)$ implies the existence of $\Gamma(k)$ for all $k \in \mathbb{Z}^m$. However, we will now demonstrate that the existence of all $\Gamma(k)$ only depends on the existence of $\Gamma(0)$.

Theorem 3.4. *If* $\Gamma(0) = \lim_{\underline{R}\Rightarrow\infty} \Gamma_{\underline{R}}(0)$ *exists, then*

$$\Gamma(k) = \lim_{\underline{R}\Rightarrow\infty} \Gamma_{\underline{R}}(k)$$

exists for all $k \in \mathbb{Z}^m$.

Proof. Let $U = B_{r^*}(0) \cap \mathbb{Z}^m$, where r^* is given by equation (22). The collection of vectors $\Gamma_{\underline{R}}(k)$ with $k \in U$ is considered as a $(U \times (\ker(f) \times \ker(f)))$-vector written as $\Gamma_{\underline{R}}$, i.e., $\Gamma_{\underline{R}}(k, g_1, g_2)$ is the value of the $(\ker(f) \times \ker(f))$-vector $\Gamma_{\underline{R}}(k)$ at the entry (g_1, g_2).

Due to equation (17) and lemma 3.3, there exists a matrix

$$B \in \mathbb{C}^{(U\times(\ker(f)\times\ker(f)))\times(U\times(\ker(f)\times\ker(f)))}$$

such that for the vector $\Gamma_{\underline{R}}$ the equation

$$\Gamma_{\underline{R}} = B\Gamma_{\underline{R}/\underline{c}} + \epsilon(\underline{R}) \tag{23}$$

holds. To emphasize the role of the component $\Gamma_{\underline{R}}(0)$ of $\Gamma_{\underline{R}}$, we write $\Gamma_{\underline{R}} = (\Gamma^*_{\underline{R}}, \Gamma_{\underline{R}}(0))$, where $\Gamma^*_{\underline{R}}$ is considered as a $(U^* \times (\ker(f) \times \ker(f)))$-vector with $U^* = U\backslash\{0\}$.

Since, according to (21),

$$\Gamma_{\underline{R}}(0) = \frac{1}{|\det(H)|}\left(\sum_{v\in V} A_v \otimes A_v\right)\Gamma_{\underline{R}/\underline{c}}(0) + \epsilon(\underline{R}),$$

one can write equation (23) as

$$\begin{pmatrix} (\Gamma_{\underline{R}}(u))_{u\in U^*} \\ \Gamma_{\underline{R}}(0) \end{pmatrix} = \begin{pmatrix} \tilde{B} & C \\ 0 & A \end{pmatrix}\begin{pmatrix} (\Gamma_{\underline{R}/\underline{c}}(u))_{u\in U^*} \\ \Gamma_{\underline{R}/\underline{c}}(0) \end{pmatrix} + \epsilon(\underline{R}), \tag{24}$$

where $A = \frac{1}{|\det(H)|}\left(\sum_{v\in V} A_v \otimes A_v\right)$ and with proper matrices $\tilde{B}, C$.

Due to the existence of $\Gamma(0)$, one has that $\Gamma_{\underline{R}}(0) = \Gamma(0) + \delta(\underline{R})$ with $\lim_{\underline{R}\Rightarrow\infty} \delta(\underline{R}) = 0$. This gives the equation

$$(\Gamma_{\underline{R}}(u))_{u\in U^*} = \tilde{B}(\Gamma_{\underline{R}/\underline{c}}(u))_{u\in U^*} + C\Gamma(0) + C\delta(\underline{R}/\underline{c}) + \epsilon(\underline{R}). \tag{25}$$

The proof is complete if theorem 3.7 of appendix A is applicable. To this end, and referring to appendix A, the Banach space X is considered to be the vector space $\mathbb{C}^{U^*}$, the function $\mathsf{f} : \mathbb{R}^N_{\geqslant 0} \to X$ is given as $\mathsf{f}(\underline{R}) = (\Gamma_{\underline{R}}(u))_{u\in U^*}$, which is by its definition bounded on bounded sets. The map $\mathsf{A} : X \to X$ from appendix A is then given as

$$\mathsf{A}x = \tilde{B}x + C\Gamma(0),$$

and the error term figuring in equation (A.1) corresponds to $C\delta(\underline{R}/\underline{c}) + \epsilon(\underline{R})$. In other words, if one can show that the map A is a contraction, which is implied by the contraction property of matrix $\tilde{B}$, then the limit $\lim_{\underline{R}\Rightarrow\infty}(\Gamma_{\underline{R}}(u))_{u\in U^*}$ does exist. To investigate the contraction of $\tilde{B}$, we first analyse the structure of the matrix B.

Equation (17) with $Hk + w$ replaced by k transforms into

$$\Gamma_{\underline{R}}(k) = \frac{1}{|\det(H)|}\sum_{v\in W}(A_v \otimes A_{\zeta(\zeta(k)+v)})\Gamma_{\underline{R}/\underline{c}}(\kappa(k) + \kappa(\zeta(k) + v) + \epsilon(\underline{R}). \tag{26}$$

For $k \in U$ we set

$$\mathcal{M}(k) = \{\kappa(k) + \kappa(\zeta(k) + v) \mid v \in W\},$$

i.e., $\mathcal{M}(k)$ is the set of all those arguments j for $\Gamma_{\underline{R/c}}(j)$ that are needed to determine $\Gamma_{\underline{R}}(k)$ in equation (26). Due to lemma 3.3, one has that $\mathcal{M}(k) \subseteq U$ for all $k \in U$. Furthermore, for $k \in U$ and $l \in \mathcal{M}(k)$ we set

$$\Lambda(k,l) = \{v \mid l = \kappa(k) + \kappa(\zeta(k) + v),\, v \in W\}.$$

If one associates with $k \in U$ the vector space $V_k = \mathbb{C}^{\ker(f)\times\ker(f)}$, which is the vector space where the vectors $\Gamma_{\underline{R}}(k)$ belong to, then one sees that equation (26) induces a linear map $A_{k,l}$ (different from zero) from V_l to V_k if and only if $l \in \mathcal{M}(k)$. Moreover, the matrix of this linear map is given as

$$A_{k,l} = \frac{1}{|\det(H)|} \sum_{v\in\Lambda(k,l)} A_v \otimes A_{\zeta(\zeta(k)+v))}, \tag{27}$$

which is a nonnegative matrix with at least one positive entry in every row. For reasons of convenience, we set $A_{k,l} = 0$ if $l \notin \mathcal{M}(k)$. The relation $l \in \mathcal{M}(k)$ is encoded in a matrix $G = (g_{k,l})_{k,l\in U} \in \{0,1\}^{U\times U}$ by defining

$$g_{k,l} = \begin{cases} 1 & \text{if } l \in \mathcal{M}(k) \\ 0 & \text{otherwise.} \end{cases}$$

Then G can be considered as the adjacency matrix of the directed graph with vertices U and a directed edge from l to k if and only if $l \in \mathcal{M}(k)$.

This shows that the matrix B can be considered as a $U \times U$ block matrix having the matrix $A_{k,l}$ as entry at the position k, l. I.e., B is a $(U \times \ker(f)\ker(f)) \times (U \times \ker(f)\ker(f))$-matrix with

$$B((k, g_1, g_2), (l, h_1, h_2)) = A_{k,l}((g_1, g_2), (h_1, h_2)). \tag{28}$$

In particular, the entries of B are nonnegative and bounded by 1, because of (27) combined with lemma 3.2 (ii). Moreover, one has

$$\sum_{(l,h_1,h_2)\in U\times\ker(f)\times\ker(f)} B((k, g_1, g_2), (l, h_1, h_2)) = 1$$

for all $(k, g_1, g_2) \in U \times \ker(f) \times \ker(f)$. The proof of this equality is as follows: by equation (28) the above sum is equal to

$$\sum_{(l,h_1,h_2)\in U\times\ker(f)\times\ker(f)} A_{k,l}((g_1, g_2), (h_1, h_2)) = \sum_{l\in\mathcal{M}(k),h_1,h_2\in\ker(f)} A_{k,l}((g_1, g_2), (h_1, h_2)),$$

since all $A_{k,l} \neq 0$ if and only if $l \in \mathcal{M}(k)$. According to equation (27) this can be written as

$$\frac{1}{|\det(H)|} \sum_{h_1,h_2\in\ker(f)} \sum_{l\in\mathcal{M}(k)} \sum_{v\in\Lambda(k,l)} (A_v \otimes A_{\zeta(\zeta(k)+v)})((g_1, g_2), (h_1, h_2)).$$

Since for every $v \in W$ there exists an $l \in \mathcal{M}(k)$, the sums over $l \in \mathcal{M}(k)$ and $v \in \Lambda(k,l)$ can be replaced by the sum over $v \in W$, this gives

$$\frac{1}{|\det(H)|} \sum_{h_1,h_2\in\ker(f)} \sum_{v\in W} (A_v \otimes A_{\zeta(\zeta(k)+v)})((g_1, g_2), (h_1, h_2)),$$

and the assertion follows from lemma 3.2 (ii). In other words, if one thinks of B as a (large) matrix, then the sum over every row is equal to 1. Therefore, every power of B has the same property.

Since B has the above-mentioned blockstructure with blocks $A_{k,l}$ one has for B^n at position $(k,l) \in U \times U$ the block

$$B^n(k,l) = \sum_{x_1,\ldots,x_{n-1}\in U} A_{k,x_1} A_{x_1,x_2} A_{x_2,x_3} \ldots A_{x_{n-1},l}.$$

Now note that since the matrices $A_{k,l}$ are either zero or have at least one positive element in each row, their products are also either zero or have at least one positive element in each row.

Using the interpretation of G as adjacency matrix of the above-mentioned directed graph, we can say that $B^n(k,l) = 0$ if and only if there is no directed path of length n from l to k in this directed graph. Indeed, if there is no directed path, then all products in the above sum are zero, and if there is a directed path of length n, then at least one product is different from zero. Moreover, this product contains in each row at least one positive entry, and in particular $B^n(k,l)$ has in each row a positive entry, if there exists a path of length n from l to k.

Now let $k \in U^*$, then one has $\kappa(k) \in \mathcal{M}(k)$. Since W is a complete digit set, it follows that there exists an $n_0 \in \mathbb{N}$ such that $\kappa^{n_0}(k) = 0$ for all $k \in U^*$. This shows that for every $k \in U^*$ there exists a path of length n_0 from 0 to k, and therefore that the matrix $B^{n_0}(k,0)$ has for every $k \in U^*$ a positive entry in every row.

Iterating equation (24), one obtains

$$\begin{pmatrix} (\Gamma_{\underline{R}}(u))_{u\in U^*} \\ \Gamma_{\underline{R}}(0) \end{pmatrix} = \begin{pmatrix} \tilde{B}^{n_0} & C_{n_0} \\ 0 & A^{n_0} \end{pmatrix} \begin{pmatrix} (\Gamma_{\underline{R}/\underline{c}^{n_0}}(u))_{u\in U^*} \\ \Gamma_{\underline{R}/\underline{c}^{n_0}}(0) \end{pmatrix} + \epsilon_{n_0}(\underline{R}).$$

As we have seen, the row sum of every row of B^{n_0} is equal to 1. Since the columns formed by C_{n_0} correspond to the matrices $B^{n_0}(k,0)$ which contain at least one positive entry in each row, it follows that the row sum of every row of $\tilde{B}^{n_0}$, which has nonnegative entries, is strictly less than 1. Therefore, the eigenvalues of $\tilde{B}^{n_0}$ have modulus less than 1. This implies that the eigenvalues of $\tilde{B}$ have modulus less than 1. In other words, $\tilde{B}$ is a contraction, and equation (25), with $\lim \epsilon(\underline{R}) + C\delta(\underline{R}/\underline{c}) = 0$ satisfies the assumptions of theorem 3.7. This implies that $(\Gamma(u))_{u\in U^*} = \lim_{\underline{R}\Rightarrow\infty}(\Gamma_{\underline{R}/\underline{c}^{n_0}}(u))_{u\in U^*}$ exists. $\square$

As theorem 3.7 applies to the recursion equation (25), one has

Corollary 3.5. $(\Gamma(u))_{u\in U^*}$ *is the unique solution of the equation*

$$x = \tilde{B}x + C\Gamma(0). \tag{29}$$

The unique solution of equation (29) gives the correlation function $\Gamma(k)$ for $k \in U^* = (B_{r^*}(0) \cap \mathbb{Z}^m)\backslash\{0\}$, if $\Gamma(0)$ is known. Note that the quantities $\tilde{B}, C, r^*, U^*$ can be obtained from the quantities H, W, A_w^f that define f. Then, as a consequence of lemma 3.3, it is possible to calculate $\Gamma(k)$ for all $k \in \mathbb{Z}^m$, using equation (21). The only remaining and difficult problem is the determination of $\Gamma(0)$, which involves the determination of the limits $\frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))}\sum_{x\in P\mathcal{C}(\underline{R})\cap\mathbb{Z}^m} g(x)\overline{h}(x)$ for $g, h \in \ker(f)$, and depends on the particular case under consideration.

Remarks

(i) As a consequence of the fact that $\tilde{B}$ is a contraction, the solution of equation (29) can be obtained by iterating the map $x \mapsto \tilde{B}x + C\Gamma(0)$. Observe that the size of matrix $\tilde{B}$ in equation (29) equals $|\ker(f)|^2|U^*|$, and that it can become very large, even for the most simple automatic sequences (see the next example). However, for practical purposes, it is often possible to decrease the size of the problem by taking into account the following. Reconsider the graph defined by the relation $l \in \mathcal{M}(k)$ with $l, k \in U$. A subset U' of U defines a strongly connected component of the graph if and only if for $k, l \in U'$ there exists a path from l to k and a path from k to l. By $\tilde{U}$ we denote the union of all vertices of the strongly connected components. If $k \in U\backslash\tilde{U}$, then there exist paths from k to $\tilde{U}$. With regard to the equation $x^* = \tilde{B}x^* + C\Gamma(0)$, this means that $\Gamma(k)$ is completely determined by certain values of $\Gamma(u)$ with $u \in \tilde{U}$. If $x_{\tilde{U}}, \tilde{B}_{\tilde{U}}, C_{\tilde{U}}$ denote the restriction on $\tilde{U}$, then the solution of

$$x_{\tilde{U}} = \tilde{B}_{\tilde{U}} x_{\tilde{U}} + C_{\tilde{U}} \Gamma(0) \tag{30}$$

already determines all other values of $\Gamma(k)$.

(ii) If $\Gamma(0)$ exists, then note that it follows from equation (18), with $k = 0$ and $w = 0$, that

$$\Gamma(0) = \frac{1}{|\det(H)|} \left(\sum_{v \in W} A_v \otimes A_v \right) \Gamma(0).$$

This means that $\Gamma(0)$ has to be a fixed point of the matrix $\frac{1}{|\det(H)|} \sum_{v \in W} A_v \otimes A_v$. A necessary condition for this is that this matrix has eigenvalue 1. This is the case, as follows from the fact that the sum of the entries in every row is equal to 1 (as a consequence of lemma 3.2 (i)).

Example. Consider, as a generalization of the two-dimensional Thue–Morse sequence defined in [3], the m-dimensional Thue–Morse sequence t. It is defined for some expanding matrix H with $|\det(H)| = 2$ which has the property of being similar to some block-diagonal matrix Λ satisfying equations (10) and (11). Let $W = \{0, w\}$ be a complete digit set of H. The decimation matrices are given by

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where rows and columns correspond to the two kernel elements t, i.e., the Thue–Morse sequence, and g, in that order. Let $\mathcal{F}(0) = (1, -1)^T$, where ^{T} means transpose. Then $\mathcal{F}(0)$ is a fixed point of A_0. Let $\mathcal{F}(x)$ be defined as in equation (5), then $\mathcal{F}(x) = (t(x), g(x))^T$. Assuming that the limits $\Gamma_{\underline{R}}(k)$ exist, we write $\Gamma(k) = (\gamma_{tt}(k), \gamma_{tg}(k), \gamma_{gt}(k), \gamma_{gg}(k))^T$.

By theorem 3.4, we have to start with $\Gamma(0)$. Since $\mathcal{F}(0) = (1, -1)^T$, and due to its definition, the possible $\mathcal{F}(x)$-values are given as $\Pi \mathcal{F}(0)$, where Π is any possible product of the two decimation matrices A_0 and A_w. Now it is easy to see that Π only equals A_0 or A_w. As a consequence, one has that $(t(x), g(x))$ is either $(1, -1)$ or $(-1, 1)$. This allows us to conclude that

$$\gamma_{tt}(0) = \lim_{\underline{R} \Rightarrow \infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} t(x) t(x) = 1.$$

and that $\gamma_{gg}(0) = 1$ and $\gamma_{tg}(0) = \gamma_{gt}(0) = -1$. By theorem 3.4, the correlations $\Gamma(k)$ then exist for all $k \in \mathbb{Z}^m$. They can be computed by first determining $\Gamma(k)$ for $k \in U = B_{r^*} \cap \mathbb{Z}^m$, where r^* is as in lemma 3.3, as the unique solution of equation (29).

We will illustrate this for the two-dimensional case with $H = \left(\begin{smallmatrix} -1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and the complete digit set $W = \{0, w\} = \{(0,0)^T, (1,0)^T\}$. Observe that $\kappa(x) = H^{-1}x$ and $\zeta(x) = 0$ if $x \in H\mathbb{Z}^2$, and $\kappa(x) = H^{-1}(x - w)$ and $\zeta(x) = w$ if $x \in H\mathbb{Z}^2 + w$. As the expansion constant (w.r.t. the Euclidian norm) of H equals $\sqrt{2}$, then the constants α, β in lemma 3.3 are $\alpha = \sqrt{2}$ and $\beta = 1$. Thus we find that $r^* = (1 + \sqrt{2})^2 / (\sqrt{2} - 1) \approx 7.2426$. The number of points of $U = B_{r^*} \cap \mathbb{Z}^2$ equals 169. Thus $|U| = 169$, and as $|\ker(f)| = 2$, it follows that Γ_U is a vector with $4 \times 169 = 676$ components, and B is a 676×676-matrix. B is formed by considering equations (19) and (20) for $k \in U$, which read here as

$$\Gamma(k) = \tfrac{1}{2}(A_0 \otimes A_0 + A_w \otimes A_w)\Gamma(H^{-1}k) \qquad \text{if} \quad k \in H\mathbb{Z}^2$$
$$\Gamma(k) = \tfrac{1}{2}(A_0 \otimes A_w)\Gamma(H^{-1}(k - w)) + \tfrac{1}{2}(A_w \otimes A_0)\Gamma(H^{-1}(k + w)) \qquad \text{if} \quad k \in H\mathbb{Z}^2 + w.$$

It means that the matrix B can be considered as a 169×169 block matrix of 4×4-matrices $\Lambda_{k,l}$, with $k, l \in U$. What one sees in 'block'-row $k \in U$ depends on k. If $k \in H\mathbb{Z}^2$, then the matrix $\Lambda_{k,H^{-1}k}$ is equal to

$$\frac{1}{2}(A_0 \otimes A_0 + A_w \otimes A_w) = R = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

and all other matrices are zero.

If $k \in H\mathbb{Z}^2 + w$, then there are two nonzero matrices, namely

$$\Lambda_{k,H^{-1}(k-w)} = \frac{1}{2}(A_0 \otimes A_w) = S = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\Lambda_{k,H^{-1}(k+w)} = \frac{1}{2}(A_w \otimes A_0) = T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The 672×672-matrix $\tilde{B}$ and the 672×4-matrix C in equation (29) are then the corresponding top-left submatrix of B and the matrix formed by the last four columns up to row 672 of B.

However, referring to remark 1, less values of $\Gamma(k)$ need to be known in order to compute the rest of them. We consider the directed graph G for this particular example. A vertex $k \in U \cap H\mathbb{Z}^2$ has one incoming edge that is connected with the vertex $H^{-1}k$. Each vertex $k \in U \cap (H\mathbb{Z}^2 + w)$ has two incoming edges that connect it with the vertices $H^{-1}(k + w)$ and $H^{-1}(k - w)$. Inspection of this graph shows that there are two strongly connected components: $\{(0, 0)^T\}$ and $\{(1, 0)^T, (0, 1)^T, (-1, 0)^T, (0, -1)^T, (1, 1)^T, (-1, -1)^T\}$. The union of these strongly connected components forms the set $\tilde{U}$ in remark 1 and consists of the points

$$\begin{array}{llll} \tilde{u}_1 = (1, 0)^T & \tilde{u}_2 = (0, 1)^T & \tilde{u}_3 = (-1, 0)^T & \tilde{u}_4 = (0, -1)^T \\ \tilde{u}_5 = (1, 1)^T & \tilde{u}_6 = (-1, -1)^T & \tilde{u}_7 = (0, 0)^T. & \end{array}$$

It means that, for $l \notin \tilde{U}$, there is no path in the graph G that starts in l and ends in $\tilde{U}$. Or: $\Gamma(l)$ for $l \notin \tilde{U}$ is determined from the values of $\Gamma(k)$, $k \in \tilde{U}$. Thus we find as the matrix $B_{\tilde{U}}$ (rows and columns ordered according to $\tilde{u}_1, \tilde{u}_2, \ldots$):

$$B_{\tilde{U}} = \begin{pmatrix} \tilde{B}_{\tilde{U}} & C_{\tilde{U}} \\ 0 & A_{\tilde{U}} \end{pmatrix} = \left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & T & S \\ S & 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S & 0 & T \\ 0 & S & T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & R \end{array}\right).$$

The simplified equation (30), i.e., $x_{\tilde{U}} = \tilde{B}_{\tilde{U}}x + C_{\tilde{U}}\Gamma(0)$, can be solved because $\Gamma(0) = (1, -1, -1, 1)^T$. This ultimately yields

$$\Gamma((0, 0)^T) = (1 \quad -1 \quad -1 \quad 1)^T$$
$$\Gamma((1, 0)^T) = (-0.6 \quad 0.6 \quad 0.6 \quad -0.6)^T$$

$$\Gamma((0,1)^T) = (0.2 \quad -0.2 \quad -0.2 \quad 0.2)^T$$
$$\Gamma((-1,0)^T) = (-0.6 \quad 0.6 \quad 0.6 \quad -0.6)^T$$
$$\Gamma((0,-1)^T) = (0.2 \quad -0.2 \quad -0.2 \quad 0.2)^T$$
$$\Gamma((1,1)^T) = (0.2 \quad -0.2 \quad -0.2 \quad 0.2)^T$$
$$\Gamma((-1,-1)^T) = (0.2 \quad -0.2 \quad -0.2 \quad 0.2)^T.$$

Using these values, all the remaining values $\Gamma(k)$ can be computed.

The next theorem shows that in certain cases it is not necessary to consider $\Gamma(0)$ for all elements of the kernel $\ker(f)$ of f.

Let f be (H, W)-automatic. A subset $\mathcal{S} \subseteq \ker(f)$ is called a *sink* (of f) if

(i) $\partial_w(\mathcal{S}) \subseteq \mathcal{S}$ for all $w \in W$, i.e., S is a decimation invariant set.
(ii) For all $h \in \ker(f)$ there exist $n \in \mathbb{N}$ and $\omega_0, \ldots, \omega_n \in W$ such that $\partial_{\omega_0} \circ \cdots \circ \partial_{\omega_n}(h) \in \mathcal{S}$.

Theorem 3.6. *Let f be H-automatic, $\mathcal{S} \subseteq \ker(f)$ be a sink and $\mathcal{F}_{\mathcal{S}} = (h)_{h\in\mathcal{S}}$. If*

$$\Gamma(\mathcal{F}_{\mathcal{S}})(0) = \lim_{\underline{R}\Rightarrow\infty} \Gamma(\mathcal{F}_{\mathcal{S}})_{\underline{R}}(0)$$

exists, then

$$\Gamma(\mathcal{F})(0) = \lim_{\underline{R}\Rightarrow\infty} \Gamma(\mathcal{F})_{\underline{R}}(0)$$

exists.

Proof. By equation (26) one has for $k = 0$ and $w = 0$

$$\Gamma_{\underline{R}}(\mathcal{F})(0) = \frac{1}{|\det(H)|}\left(\sum_{v\in W} A_v \otimes A_v\right)\Gamma_{\underline{R/c}}(\mathcal{F})(0) + \epsilon(\underline{R}).$$

If one sets $\rho_{\underline{R}}(0) = (\Gamma_{\underline{R}}(0)(g,h))_{(g,h)\notin\mathcal{S}\times\mathcal{S}}$ and $\sigma_{\underline{R}}(0) = (\Gamma_{\underline{R}}(0)(g,h))_{(g,h)\in\mathcal{S}\times\mathcal{S}}$, then together with the fact that $\mathcal{S}$ is a sink, the above equation can be written as

$$\begin{pmatrix}\rho_{\underline{R}}(0)\\ \sigma_{\underline{R}}(0)\end{pmatrix} = \begin{pmatrix} B & C\\ 0 & A\end{pmatrix}\begin{pmatrix}\rho_{\underline{R/c}}(0)\\ \sigma_{\underline{R/c}}(0)\end{pmatrix} + \epsilon(\underline{R}).$$

Note that the above matrix is nonnegative and that the row sums are always equal to 1. Using the fact that $\mathcal{S}$ is a sink, it follows that there exists an $n_0 \in \mathbb{N}$ such that

$$\begin{pmatrix}\rho_{\underline{R}}(0)\\ \sigma_{\underline{R}}(0)\end{pmatrix} = \begin{pmatrix} B^{n_0} & C_{n_0}\\ 0 & A^{n_0}\end{pmatrix}\begin{pmatrix}\rho_{\underline{R/c^{n_0}}}(0)\\ \sigma_{\underline{R/c^{n_0}}}(0)\end{pmatrix} + \epsilon_{n_0}(\underline{R}),$$

such that each row of the matrix C_{n_0} contains at least one nonzero element. This shows that the row sums of the matrix B^{n_0} are always less than 1. This means that B is a contraction. Since $\lim \sigma_{\underline{R}}(0)$ exists, it follows from theorem 3.7 that $\lim \rho_{\underline{R}}(0)$ exists and therefore the limit of $\Gamma_{\underline{R}}(0)$, i.e., $\Gamma(0)$, exists. $\square$

We conclude with some additional examples. The expanding map H is supposed to have $|\det(H)| = 2$ and to be similar to a block matrix Λ, see equations (10), (11). Moreover, we assume that $W = \{0, w\}$ is a complete digit set of H. As it was done in [3] for dimension 2 it is possible to define the higher dimensional analogues of the paper folding and the Rudin–Shapiro sequence with values ± 1.

Paper folding sequence. Define $p(0) = 1$ and recursively define $p(x)$ for $x \in \mathbb{Z}^m$ by setting

$$p(H^2x) = 1 \quad p(Hx + w) = p(x) \quad p(H^2x + Hw) = -1.$$

Then the kernel consists of $p, g : g(x) = p(Hx)$ and the constant sequences $\mathbf{1}$ and $-\mathbf{1}$ ($\mathbf{1}$ denoting that all elements of $\mathbb{Z}^2$ map into 1). Moreover, the subset $\mathcal{S} = \{\pm\mathbf{1}\}$ is a sink of the kernel for which $\Gamma(\mathcal{F}_{\mathcal{S}})(0)$ exists. Therefore, by theorem 3.6, $\Gamma(0)$ exists which implies that $\Gamma(k)$ exists for all $k \in \mathbb{Z}^m$.

Rudin–Shapiro sequence. Set $r(0) = 1$ and recursively define $r(x)$ by setting

$$r(Hx) = r(x) \quad r(H^2x + w) = r(x) \quad r(H^2x + Hw + w) = -r(Hx + w).$$

Using these defining relations, one obtains $\ker(r) = \{r, g, -r, -g\}$, where $g(x) = r(Hx + w)$. The relevant components of $\mathcal{F}_r \otimes \mathcal{F}_r$ are the sequences $r^2 = (r(x)^2)_{x\in\mathbb{Z}^m}, rg = (r(x)g(x))_{x\in\mathbb{Z}^m}$, and $g^2 = (g(x)g(x))_{x\in\mathbb{Z}^m}$. Note that r^2 and g^2 are the constant sequence $\mathbf{1}$. It follows that $\gamma_{xx}(0) = 1$ and $\gamma_{x(-x)} = -1$ for $x \in \ker(r)$. Since $\partial_0(rg) = r^2 = \mathbf{1}$ and $\partial_w(rg) = -gg = -\mathbf{1}$, i.e., $rg(x) = 1$ if $x \in H\mathbb{Z}^m$ and $rg(x) = -1$ if $x \in H\mathbb{Z}^m + w$, it follows that

$$\gamma_{rg}(0) = \lim_{\underline{R}\Rightarrow\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x\in P\mathcal{C}(\underline{R})\cap\mathbb{Z}^m} r(x)g(x) = 0.$$

For the same reasons $\gamma_{xy}(0) = 0$ for all $x, y \in \ker(r), x \neq y$. Thus the limits defining $\Gamma(0)$ all exist, which implies the existence of $\Gamma(k)$ for all $k \in \mathbb{Z}^m$.

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Appendix A. An analytical result

In this section, we state and prove the theorem which was used in the proofs of theorems 3.4 and 3.6.

Let X be a Banach space, i.e., a real/complex, normed, complete vector space. Furthermore, let $\mathsf{A} : X \to X$ be a contraction, i.e., there exists $0 \leqslant \lambda < 1$ such that

$$\|\mathsf{A}x - \mathsf{A}y\| \leqslant \lambda\|x - y\|$$

for all $x, y \in X$. Note that, due to Banach's fixed point theorem, there exists a unique $x^* \in X$ with $\mathsf{A}x^* = x^*$.

Let $\mathsf{f} : \mathbb{R}^n_{\geqslant 0} \to X$ be a function which is bounded on every bounded subset of $\mathbb{R}^n_{\geqslant 0}$ and let $\underline{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$ such that $c_i > 1$ for $i = 1, \dots, n$. Moreover, assume that f satisfies

$$\mathsf{f}(\underline{R}) = \mathsf{Af}\left(\frac{\underline{R}}{\underline{c}}\right) + \epsilon(\underline{R}), \tag{A.1}$$

for all $\underline{R} = (R_1, \dots, R_n), R_i \geqslant 0, i = 1, \dots, n$, where $\underline{R}/\underline{c} = (R_1/c_1, \dots, R_n/c_n)$, and assume that

$$\lim_{\underline{R}\Rightarrow\infty} \epsilon(\underline{R}) = 0,$$

where $\underline{R} \Rightarrow \infty$ means that every component tends to infinity. Due to the boundedness of f on bounded subsets and the limit condition on ϵ, it follows that ϵ is bounded, i.e.,

$$|\epsilon(\underline{R})| \leqslant K_1.$$

We also agree that for $s \in \mathbb{N}$ the notion $\underline{c}^s$ means $\left(c_1^s, \dots, c_n^s\right)$.

Theorem 3.7. *Under the above assumptions*

$$\lim_{\underline{R}\Rightarrow\infty} \mathsf{f}(\underline{R}) = x^*.$$

Proof. Define $g(\underline{R}) = \mathsf{f}(\underline{R}) - x^*$, then an easy computation shows that

$$g(\underline{R}) = \mathsf{B}g\left(\frac{\underline{R}}{\underline{c}}\right) + \epsilon(\underline{R}), \tag{A.2}$$

with $\mathsf{B} : X \to X$ defined as $\mathsf{B}x = \mathsf{A}(x + x^*) - x^*$. Note further that B is a contraction with $\|\mathsf{B}x - \mathsf{B}y\| \leqslant \lambda\|x - y\|$ with the unique fixed point 0. Thus, the assertion is proved, if $\lim_{\underline{R}\Rightarrow\infty} g(\underline{R}) = 0$ is established.

A repeated application of equation (A.2) in connection with the triangle inequality leads to

$$\|g(\underline{R})\| \leqslant \lambda^{s+1}\left\|g\left(\frac{\underline{R}}{\underline{c}^{s+1}}\right)\right\| + \sum_{j=0}^{s}\lambda^j\left\|\epsilon\left(\frac{\underline{R}}{\underline{c}^j}\right)\right\| \tag{A.3}$$

for all $\underline{R}$ and all $s \in \mathbb{N}$.

Let $\delta > 0$ be given. Since $\lim_{\underline{R}\Rightarrow\infty}\epsilon(\underline{R}) = 0$, there exists an $M_1 > 1$ such that

$$|\epsilon(\underline{R})| < \frac{\delta}{3K_2} \tag{A.4}$$

for all $\underline{R} > M_1$, i.e., $R_i > M_1$ for $i = 1, \ldots, n$, and where $K_2 = \sum_{j=0}^{\infty}\lambda^j$.

Set $c^+ = \max\{c_i \mid i = 1, \ldots, n\}$ and for

$$\underline{R} > M_1^{1+\frac{\log c^+}{\log M_1}} \tag{A.5}$$

set

$$\begin{aligned} s(\underline{R}) &= \left\lceil \max\left\{\frac{\log R_i}{\log c_i}\,\middle|\, i = 1, \ldots, n\right\}\right\rceil \\ t(\underline{R}) &= \left\lfloor \min\left\{\frac{\log R_i - \log M_1}{\log c_i}\,\middle|\, i = 1, \ldots, n\right\}\right\rfloor, \end{aligned} \tag{A.6}$$

where $\lceil u\rceil$ is the smallest integer greater than or equal to u, and $\lfloor u\rfloor$ is the largest integer smaller than or equal to u.

Then one has $s(\underline{R}) > t(\underline{R}) \geqslant 1$, the rightmost inequality being a consequence of equation (A.5). According to the definition of $s(\underline{R})$, one has

$$0 \leqslant \frac{\underline{R}}{\underline{c}^{s(\underline{R})+1}} \leqslant 1$$

for all $\underline{R}$ satisfying equation (A.5). Combined with the fact that g is bounded on bounded subsets, it follows that

$$\lambda^{s(\underline{R})+1}\left\|g\left(\frac{\underline{R}}{\underline{c}^{s(\underline{R})+1}}\right)\right\| \leqslant \lambda^{s(\underline{R})+1}K_3 \tag{A.7}$$

for these $\underline{R}$ and an appropriate $K_3 > 0$.

The sum in equation (A.3) is split into two parts, namely

$$\sum_{j=t(\underline{R})+1}^{s(\underline{R})}\lambda^j\left\|\epsilon\left(\frac{\underline{R}}{\underline{c}^j}\right)\right\|,$$

using the fact that $|\epsilon(\underline{R})| \leqslant K_1$ this sum becomes

$$K_1 \sum_{j=t(\underline{R})+1}^{s(\underline{R})} \lambda^j \leqslant K_1 K_2 \lambda^{t(\underline{R})+1}, \tag{A.8}$$

due to the definition of K_2 above.

The second part, namely

$$\sum_{j=0}^{t(\underline{R})} \lambda^j \left\| \epsilon\left(\frac{\underline{R}}{\underline{c}^j}\right) \right\|,$$

can be bounded as follows. Since $\underline{R}$ satisfies inequality (A.5) and due to the definition, see (A.6), of $t(\underline{R})$, one easily sees that

$$\frac{\underline{R}}{\underline{c}^j} > M_1$$

holds for all $j = 0, \ldots, t(\underline{R})$. Therefore, according to equation (A.4) and the definition of K_2, one has

$$\sum_{j=0}^{t(\underline{R})} \lambda^j \left\| \epsilon\left(\frac{\underline{R}}{\underline{c}^j}\right) \right\| \leqslant \frac{\delta}{3K_2} \sum_{j=0}^{t(\underline{R})} \lambda_j \leqslant \frac{\delta}{3}. \tag{A.9}$$

Putting equations (A.7), (A.8) and (A.9) together, one obtains

$$\|g(\underline{R})\| \leqslant \lambda^{s(\underline{R})+1} K_3 + \lambda^{t(\underline{R})+1} K_1 K_2 + \frac{\delta}{3}$$

for all $\underline{R}$ satisfying (A.5). Due to the definition of $s(\underline{R})$ and $t(\underline{R})$ and the fact that $0 \leqslant \lambda < 1$, there exists M_2 such that for all $\underline{R} > M_2$

$$|\lambda^{s(\underline{R})+1} K_3| < \frac{\delta}{3} \qquad \text{and} \qquad |\lambda^{t(\underline{R})+1} K_1 K_2| < \frac{\delta}{3}.$$

Setting $M = \max\left\{M_1^{1+\frac{\log c^+}{\log M_1}}, M_2\right\}$, one has

$$\|g(\underline{R})\| < \delta$$

for all $\underline{R} > M$. This completes the proof. □

Appendix B. A geometric property

Let H be an expanding $m \times m$ integer matrix for which there exists a matrix $P \in \mathbb{R}^{m\times m}$ such that $\Lambda = P^{-1}HP$ is a proper block-diagonal matrix of the form

$$\Lambda = P^{-1}HP = \operatorname{diag}(\lambda_1, \ldots, \lambda_s, \Lambda_1, \ldots, \Lambda_t),$$

as described in the introduction (equations (10) and (11)). Let $\underline{R}$ be an $(s+t)$-vector with positive real entries. Then the cylinder $\mathcal{C}(\underline{R}) \subset \mathbb{R}^m$ is defined as the set

$$\begin{aligned} \mathcal{C}(\underline{R}) = \{(x_1, \ldots, x_s) \mid |x_i| \leqslant R_i, i = 1, \ldots, s\} \\ \times\left\{(x_1, y_1, \ldots, x_t, y_t) \,\middle|\, x_j^2 + y_j^2 \leqslant R_{s+j}^2, j = 1, \ldots, t\right\}. \end{aligned} \tag{B.1}$$

Using this notation one has

$$\Lambda^{-1}(\mathcal{C}(\underline{R})) = \mathcal{C}(\underline{R}/\underline{c}), \tag{B.2}$$

where $\underline{R}/\underline{c} = (R_1/c_1, \ldots, R_{s+t}/c_{s+t})$ with $c_i = |\lambda_i|$ for $i = 1, \ldots, s$ and $c_{s+j} = |\det(\Lambda_j)|^{\frac{1}{2}}$ for $j = 1, \ldots, t$. The volume of the cylinder is given by

$$\mathrm{vol}(\mathcal{C}(\underline{R})) = 2^s \pi^t \prod_{i=1}^{s} R_i \prod_{j=1}^{t} R_{s+j}^2. \tag{B.3}$$

We will need to consider limits where $\mathrm{vol}(\mathcal{C}(\underline{R})) \to \infty$ by letting the $R_i \to \infty$ simultaneously, which we denote by $\underline{R} \Rightarrow \infty$. If $E : \mathbb{R}_{\geqslant 0}^{s+t} \to \mathbb{C}$ is such that $\lim_{\underline{R}\Rightarrow\infty} \frac{E(\underline{R})}{\mathrm{vol}(\mathcal{C}(\underline{R}))} = 0$, then we write

$$o_{\mathcal{C}}(\underline{R}) \tag{B.4}$$

for E.

Theorem 3.8. *For* $\underline{R} = (R_1, R_2, \ldots, R_{s+t}) \in \mathbb{R}_{\geqslant 0}^{s+t}$

$$|P^{-1}(\mathbb{Z}^m) \cap \mathcal{C}(\underline{R})| = |\det(P)|\, \mathrm{vol}(\mathcal{C}(\underline{R})) + o_{\mathcal{C}}(\underline{R}). \tag{B.5}$$

Proof. The proof runs along similar lines as the proof of Satz 1 in chapter 2 of [13], so we only sketch it. Let $P^{-1} = [v_1, \ldots, v_m]$, where the v_i are column vectors. For $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$, we denote an elementary cell $Z(k)$ as the set

$$Z(k) = \left\{ \sum_{i=1}^{m} (k_i + t_i) v_i \,\middle|\, 0 \leqslant t_j \leqslant 1,\ j = 1, \ldots, m \right\}$$

and $\sum_{i=1}^{m} k_i v_i$ is called the lower left corner of $Z(k)$. With $\mathcal{P}$ we denote the polytope that consists of all elementary cells $Z(k)$ whose lower left corner belongs to $\mathcal{C}(\underline{R})$. Since the diameter of an elementary cell is finite, there exists a constant $c > 0$ such that

$$\mathcal{C}(R_1 - c, \ldots, R_{s+t} - c) \subseteq \mathcal{P} \subseteq \mathcal{C}(R_1 + c, \ldots, R_{s+t} + c).$$

In terms of volumes this yields

$$\mathrm{vol}(\mathcal{C}(R_1 - c, \ldots, R_{s+t} - c)) \leqslant \mathrm{vol}(\mathcal{P}) \leqslant \mathrm{vol}(\mathcal{C}(R_1 + c, \ldots, R_{s+t} + c)).$$

Using the fact that $\mathrm{vol}(\mathcal{P}) = |\det(P^{-1})||P^{-1}(\mathbb{Z}^m) \cap \mathcal{C}(\underline{R})|$ one obtains

$$\begin{aligned}
&|\det(P)|\, (\mathrm{vol}(\mathcal{C}(R_1 - c, \ldots, R_{s+t} - c)) - \mathrm{vol}(\mathcal{C}(\underline{R})))\\
&\qquad \leqslant |P^{-1}(\mathbb{Z}^m) \cap \mathcal{C}(\underline{R})| - |\det(P)|\, \mathrm{vol}(\mathcal{C}(\underline{R}))\\
&\qquad \leqslant |\det(P)|\, (\mathrm{vol}(\mathcal{C}(R_1 + c, \ldots, R_{s+t} + c)) - \mathrm{vol}(\mathcal{C}(\underline{R}))).
\end{aligned}$$

Since $\mathrm{vol}(\mathcal{C}(\underline{R}))$ is a polynomial in the variables R_i see equation (B.3), it follows that the differences on the left- and right-hand side of the inequality are $o_{\mathcal{C}}(\underline{R})$. □

Corollary 3.9. *When* $P\mathcal{C}(\underline{R})$ *denotes the transformation of the cylinder* $\mathcal{C}(\underline{R})$ *under* P*, then equation (B.5) is equivalent to*

$$|\mathbb{Z}^m \cap P\mathcal{C}(\underline{R})| = \mathrm{vol}(P\mathcal{C}(\underline{R})) + o_{\mathcal{C}}(\underline{R}).$$

An important consequence of the above theorem is a kind of summation formula. As in appendix A, X denotes a Banach space. To state the summation formula we assume that $F : \mathbb{Z}^m \to X$ is any bounded function, i.e., $\|F(x)\| \leqslant M$ for all $x \in \mathbb{Z}^m$. Furthermore, let W be any residue set for H.

Lemma 3.10. *Let* $\underline{R} \in \mathbb{R}_{+}^{s+t}$ *then*

$$\sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} F(x) = \sum_{v \in W} \sum_{x \in P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m} F(Hx + v) + o_{\mathcal{C}}(\underline{R}),$$

where $\underline{R}/\underline{c}$ *is as in equation (B.2).*

Proof. Note that $P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m \subset P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m$. Therefore, there are two possibilities for a difference in the left and in the right sum. The first way is that $Hx + v$ does not belong to $P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m$ for an $x \in P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m$ and a $v \in V$, the second way is that for $x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m$ there exists no $y \in P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m$ and no $v \in V$ such that $x = Hy + v$. Since the diameter of the residue set W is finite, these errors can only occur in a proximity of the boundary of $\mathcal{C}(\underline{R})$. In other words, there exists $c > 0$ such that

$$\left\| \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} F(x) - \sum_{v \in W} \sum_{x \in P\mathcal{C}(\underline{R}/\underline{c}) \cap \mathbb{Z}^m} F(Hx + w) \right\| \leqslant M \| (P\mathcal{C}(R_1 + c, \ldots, R_{s+t} + c) \backslash P\mathcal{C}(R_1 - c, \ldots, R_{s+t} - c)) \cap \mathbb{Z}^m \|.$$

By using corollary 3.9, one obtains the result. □

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